STRUCTURE OF REGULAR AND IDEMPOTENT TERNARY SEMIHYPER RINGS

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ABSTRACT:
In this paper we have studied the nature of sets and elements of “ternary semihyperring” with variety of examples and results. The relation between “regular elements and idempotent elements of a ternary semihyperring” has been developed. The notion of “hyper ideals of a ternary semihyperring” could characterize the regularity conditions of “ternary semihyperring”.

Keywords: “Semihyperring, Ternary semihyperring”, “Hyper ideal(HI)”, “Hyper idempotent”, “sub-hyper idempotent”.

I. INTRODUCTION:
“Ternary Semigroup” theory was studied by SIOSON in the year 1965. After that he studied about “regular ternary semi groups”. DIXIT and DEWAN in the year 1995 introduced the notion of ideals in “ternary semi groups” In 1934 the hyper structures was introduced further the structures of “hyper groups” was studied by MARTY.

II. PRELIMINARIES:
Def 2.1: A non empty set H with two “operations” such as “+, [ ]” is known as “ternary semi ring” if H is an “additive commutative semi group” with the following axioms

1. \[ [h_1 h_2 h_3] h_4 h_5] = [h_1 [h_2 h_3 h_4] h_5] = [h_1 h_2 [h_3 h_4 h_5]]
2. \[ (h_1 + h_2) h_3 h_4] = [h_1 h_3 h_4] + [h_2 h_3 h_4]
3. \[ h_1 (h_2 + h_3) h_4] = [h_1 h_2 h_4] + [h_1 h_3 h_4]
4. \[ h_1 h_2 (h_3 + h_4)] = [h_1 h_2 h_3] + [h_1 h_2 h_4] \forall h_1, h_2, h_3, h_4, h_5 \in H

A mapping [ ]:H × H × H → P*(H) is known as “ternary hyper operation” on the set H. Where H is a non empty subset of \( P^*(H) = P(H)\setminus\{0\} \). A “ternary hyper grouped” is known as the pair (H,[ ] ) if \( H_1, H_2, H_3 \) are the non-empty subsets of H then we define \([H_1 H_2 H_3] = \bigcup_{h_4 \in H_1, h_5 \in H_2, h_6 \in H_3} [h_4 h_5 h_6] \). A non empty set H is known as “ternary semi hyper ring” if for all \( h_1, h_2, h_3, h_4, h_5 \in H \) and (H,⊕ ) is a “commutative semi hyper group and the ternary multiplication [ ]” satisfies the following conditions
1. \[[h_1h_2h_3]h_4h_5] = [h_1h_2h_3h_4h_5] = [h_1h_2h_3]h_4h_5
2. \[(h_1 \oplus h_2)h_3 = [h_1h_2]h_3 = h_1(h_2h_3)
3. [h_1(h_2 \oplus h_3)h_4] = [h_1h_2h_3]h_4
4. [h_1h_2(h_3 \oplus h_4)] = [h_1h_2h_3]h_4

Def 2.3: A “ternary semi hyper ring is called commutative” if
\[h_1h_2h_3 = [h_2h_3h_1] = [h_3h_2h_1] = [h_2h_3h_1] \forall h_1, h_2, h_3 \in H\]

Let H be a “ternary semi hyper ring” and G is a non-empty subset of H then G is a ternary sub semi hyper ring of H if and only if \[GGG \subseteq G\]. A “ternary semihyperring” H is said to have a zero element if there exist an element 0 \(0 \in H\) such that for all \(h_1, h_2 \in H\), \([0h_1h_2] = [h_10h_2] = [h_1h_20] = 0\). An element e of “ternary semihyperring” H is called an “identity” if \([h_1he] = [h_1eh_1] = [eh_1h_1] = h_1\) for all \(h_1 \in H\) and it is clear that \([eeh_1] = [eh_1e] = [h_1ee] = h_1\).  

III. “HYPER IDEALS IN TERNARY SEMI HYPER RINGS”

Throughout this paper “left hyper ideal”, “lateral hyper ideal”, “right hyper ideal” and “hyper ideals” are denoted as “LHI, MHI, RHI and HI” respectively and H denotes the ternary semi hyper ring.

Def 3.1: A non empty “additive sub semi hyper group I of H” is said to be

1. A LHI of H if \([HHH] \subseteq I\)
2. A MHI of H if \([HHI] \subseteq I\)
3. A RHI of H if \([HII] \subseteq I\)

If I is both LHI as well as RHI of H, then I is called a two sided “HI” of H. If I is a LHI, a MHI, a RHI of H then I is known as “HI” of H.

Lemma 3.2: The “Union” of any two “HI of H” is a “HI” of H.

Remark 3.3: Let (H, \(\ominus\), [ ]) be a “ternary semi hyper ring” for every element \(h \in H\) then the left, lateral, right, two sided and “hyper ideal generated” by h are respectively shown by

\[L(h) = [h] = \{ h \} \cup [HHh] \]

\[M(r) = [r] = \{ r \} \cup [HrH] \cup [H[HrH]H] \]

\[R(r) = [r] = \{ r \} \cup [rHH] \]

\[T(r) = [r] = \{ r \} \cup [HHR] \cup [rHH] \cup [H[HrH]H] \]

\[J(r) = [r] = \{ r \} \cup [HHR] \cup [HrH] \cup [H[HrH]H] \cup [rHH] \]

Def 3.4: Let M be a “hyper ideal” of H then M is known as “maximal HI” of H if M \(\neq H\) and their does not exist any “proper HI” of I of H such that M \(\subseteq I\).

IV. "REGULAR TERNARY SEMIHYPERRING":

Def 4.1: A HI I of a H is called an “idempotent” if \([III] = I\)

Def 4.2: An element a of H is known as “hyper additive idempotent” if \(a \oplus a = \{a\}\). If \(a \ominus \{aaa\} = \{a\}\), then a is said to be “multiplicatively sub hyper idempotent”. H is called “semi simple ternary hyper semi ring” if HI of H
is “idempotent”. H is said to be a “\textit{sub-hyper idempotent ternary semihyperring}” if each element is “sub-hyper idempotent”.

**Th 4.3:** Let \(H\) satisfying the condition \(a \oplus \{aba\} \oplus a = \{a\} \forall a, b \in H\). If \(H\) contains the “ternary multiplicative identity” which is also an “hyper additive identity”, then \(T\) is a “multiplicatively sub-hyper idempotent ternary semihyperring”.

**Proof:** Since \(a \oplus \{aba\} \oplus a = \{a\}\) for all \(a, b \in H\). Let \(e\) be the “multiplicative identity” in \(H\) is also an “hyper additive identity”, i.e., \([aee] = [eae] = [eaa] = a\) and \(a \oplus e = e \oplus a = \{a\}\). Given \(a \oplus \{aba\} \oplus a = \{a\}\) for all \(a, b \in H\). Taking \(a = b\). Then \(a \oplus \{aba\} \oplus a = \{a\}\) for all \(a \in H\). Then \(H\) is a regular ternary semihyperring. 

**Proof:** Let \(H\) be an \(H\). Consider \((\oplus f) = [(\oplus f) (\oplus f) (\oplus f)] \forall f, b \in H\). Then \(H\) is “hyper additively idempotent”.

**Def 4.5:** A singleton set \{\(a\}\} of \(H\) is “regular” if \(\exists R, S \subseteq H\) such that \(\{a\} = \{a \oplus RaSa = \{x \in H \mid x = abaca, b \in R, c \in S\}\} H\) is said to be “\textit{regular ternary semihyperring}” if every element is regular.

**Ex 4.6:** If \(H = \{a, b, c, d\}\). Then \(H\) is a semihyper group w.r.t hyper operation \(\oplus\) and multiplication “.\)” as follows.

\[
\begin{array}{cccc}
  \oplus & a & b & c & d \\
  a & \{a\} & \{a, b\} & \{a, c\} & \{a, d\} \\
  b & \{a, b\} & \{b\} & \{b, c\} & \{b, c, d\} \\
  c & \{a, c\} & \{b, c\} & \{c\} & \{c, d\} \\
  d & \{a, b, c, d\} & \{b, d\} & \{c, d\} & \{d\} \\
\end{array}
\]

Then \(H\) is a “ternary semihyperring” with “ternary multiplication” \([\ ]\) as \([xyz] = x.y.z\) for \(x, y, z \in H\). Since all singleton sets are regular and hence \(H\) is a “regular ternary semihyperring”.

**Lemma 4.7:** Every “idempotent element in \(H\) is regular”.

**Proof:** Let \(r\) be an “idempotent in \(H\) \(\Rightarrow \{r\} = r \cdot r \cdot r \cdot r \cdot r\). Therefore \(r\) is regular.

**Def 4.8:** \(l \in H\). Then \(l\) is known as “left regular(lateral regular, right regular)” if \(\exists r, s \in H \ni l = [\lll]rs[l = [r\lll]s], l = [rs[\lll]]\). \(l\) is nothing but “intra regular” if \(l = [\lll]rs[l\lll]s\) and \(l\) is “completely regular” if \(i) [[rl]s]l = l, (ii) [lrl] = [rl] = [llr], (iii) \{l\} = \oplus r \ominus l, (iv) [ll(\oplus r)] = l \ominus r\). A “ternary semihyperring” \(H\) is known as “completely regular ternary semihyperring” provided every element in \(H\) is “completely regular”.

**Th 4.9:** Let \(H\) be a “ternary semihyperring” and \(a \in H\). If \(a\) is a “completely regular element”, then \(a\) is “regular, left regular, lateral regular and right regular”.

**Proof:** Suppose that \(a\) is completely regular.

Then there exist \(x, y \in H\) such that \([\lll]xa[y]a = a\]

\[\text{and} [axa] = [xax] = [aax] = [aya] = [aay] = [axy] = [yxa] = [xay] = [yax].\]
 Clearly A is regular.

Now \(a = [axa]ya = [axa]ay = [laxa]xy\). Therefore \(a\) is “left regular”.

Also \(a = [axa]ya = [axa]ya = [laxa]ay\). Therefore \(a\) is “lateral regular”.

And \(a = [axa]ya = [axa]ya = [xya]aa\). Therefore \(a\) is “right regular”.

**Th 4.10:** Let \(H\) has identity”. Then \(H\) is “regular” if and only if for any “LHI L, MHI M and RHI R” of \(H\), 
\([HIMR] = [LMR]\).

**Proof:** Since \(L\) is a LHI of \(H\). Then we have \([RML] \subseteq [HHL] \subseteq L\). Again \(M\) is a MHI of \(H\) and \(C\) is a RHI of \(H\). Hence, \([RML] \subseteq [HHL] \subseteq [LMR]\). Therefore from (1) and (2) we can write \([HIMR] = [LMR]\).

Let \(H\) be a “regular” and \(a \in LIMRI\). Since \(H\) is a “regular”, then \(\exists b, c \in H \ni a = abaca \subseteq abL \subseteq [RML]\) and hence \(LIMRI \subseteq [RML] \rightarrow (2)\). Therefore from (1) and (2) we can write \([HIMR] = [LMR]\).

In order to prove the converse part assume that \([HIMR] = [LMR], a \in H\).

Now \(L = [HHa], M = [HaH] \) and \(R = [aHH]a\). Since \(H\) is a “ternary semihyperring with identity”, all the three “HI” contains \(a\) and hence \(a \in LIMRI \subseteq [RML] = [aHH][HaH][HHa] = [aHH][aHH][a] = [aHHa]. \(\exists b, c \in H \ni a = abaca\). \(\therefore, a \in H\) is “regular element” in \(H\) and \(a\) is “arbitrary and hence \(H\) is a regular ternary semihyperring”.

**Th 4.11:** Let \(H\) be a “commutative ternary semihyperring with an identity”. Then \(H\) is “regular” if and only if \([III]\) = \([I]\) for every ideal \(I\) of \(H\).

**Proof:** Let \(H\) is regular, \(I\) is a HI of \(H\) implies that by Th 4.10, \([III]\) = \([I]\) \cap \([I]\) = \([I]\).

Conversely, let \(H\) be a “ternary semihyperring” satisfying the given condition and \(a \in H\). Then \(I = [aHH]a\) is a HI of \(H\). Since \(H\) is a “ternary semihyperring” with an “identity”, it follows that \([I] = [III] = [aHH][aHH][aHH] = [aHH][a][HH]a = [aHHa]. \(\exists b, c \in H \ni a = abaca\). \(\therefore, a \in H\) is “regular element” in \(H\) and \(a\) is “arbitrary and hence \(H\) is a regular ternary semihyperring”.

**Th 4.12:** Let \(I\) be “hyper ideal of a regular ternary semihyperring” \(H\) Then \(I\) is “regular” and any “ideal” \(J\) of \(I\) is an “hyper ideal” of \(R\).

**Proof:** Let \(I\) is “HI of a regular ternary semihyperring” \(H\) and \(a \in I \subseteq H\). Since \(H\) is regular, it follows that \(\exists b \in H \ni a = [bab]aba\). Also let \(c = [bab]ab \in I\). Then \(a = [aca]ca = [a][bab]aba[bab]aba = [(aba]ba[bab]aba\). Hence the \(“HI\) is regular”.

We now prove that if \(a \in J \subseteq I\) and \(b \in H\), then \([abh], [hab], [hha]\) are in \(J\). Let \(a[h]h \in I\). Then each \(k = [a]hh \in I\) is a “regular element” in \(I\). Hence there exist \(k_i, k_2 \in I\) such that \(k = kkk \subseteq [akhh]H \subseteq [a][H] \subseteq J\) since \(J\) is an hyper ideal of \(I\). Thus \(k = [akhh]H \subseteq J\). Hence \(a[h]h \in J\). Similar steps leads to \([a][h]h \in J, [hha] \in J\). Thus an “hyper ideal” \(J\) of \(I\) is an “hyper ideal” of \(R\).

**V. INVERSE SETS IN TERNARY SEMIHYPERRING:**

**Def 5.1:** A subset \(B\) of a “ternary semihyperring” \(H\) is said to be a “inverse” of \(A\) if \(A = [[ABA][BA]]B = [[BAB][AB]]\) denoted by \(B \in V(A)\). An element \(b\) of a “ternary semihyperring” \(H\) is known as a inverse of \(a \in H\) if \(a = [[aba][ba]], b = [[bab][ab]]\) denoted by \(b \in V(a)\). If \(B \in V(A)\), then the subsets \(A, B\) of \(H\) are “regular subsets of the ternary semihyperring \(H\)”.

**Lemma 5.2:** Let \(A\) be a “regular subset of ternary semihyperring” \(H\). Then there exists a “regular subset” \(B \in V(A)\).

**Proof:** Suppose that \(A\) is a “regular subset of a ternary semihyperring” \(H\). Then there exist \(C \subseteq H\) such that \(A = [[ACA][CA]]. Let B = [[CAC][AC]]. Then
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