GH-CLOSED SETS IN BITOPOLOGICAL SPACES

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ABSTRACT

In this paper a new class of sets namely $\eta$-closed sets, $\eta$-closure of a set, $\eta$-open sets, $\eta$-neighbourhoodsin bitopological spaces are studied and some of their basic properties are discussed.

Keywords

$\eta$-closed sets, $\eta$-closure of a set, $\eta$-open sets, $\eta$-neighbourhoods.

I. INTRODUCTION

A triplet $(X, \tau_1, \tau_2)$ where $X$ is a non-empty set and $\tau_1$ and $\tau_2$ are topologies on $X$ is called a bitopological space. In 1963, Kelly [5] initiated the study of such spaces. In 1985, Fukutake [4] introduced the concepts of g-closed sets in bitopological spaces. In this paper a new class of sets called $\eta$-closed sets, $\eta$-closure of a set, $\eta$-open sets, $\eta$-neighbourhoods in bitopological spaces and some of their basic properties are studied.

II. PRELIMINARIES

Definition : 2.1

A subset $A$ of a topological space $(X, \tau_1, \tau_2)$ is called

(i) $\tau_1\tau_2\alpha$-open set [9] if $A \subseteq \tau_1\text{int}(\tau_2\text{cl}(\tau_1\text{int}(A))) \subseteq A$.
(ii) $\tau_1\tau_2\text{pre-open set}$ [6] if $A \subseteq \tau_2\text{int}(\tau_1\text{cl}(A))$, $\tau_1\tau_2\text{pre-closed set}$ if $\tau_2\text{cl}(\tau_1\text{int}(A)) \subseteq A$.
(iii) $\tau_1\tau_2\text{semi-open set}$ [1] if $A \subseteq \tau_2\text{cl}(\tau_1\text{int}(A))$, $\tau_1\tau_2\text{semi-closed set}$ if $\tau_2\text{int}(\tau_1\text{cl}(A)) \subseteq A$.
(iv) $\tau_1\tau_2\text{regular open set}$ [2] if $A = \tau_2\text{int}(\tau_1\text{cl}(A))$, $\tau_1\tau_2\text{regular closed set}$ if $A = \tau_2\text{cl}(\tau_1\text{int}(A))$.
(v) $\tau_1\tau_2\beta$-open (or semi pre open) set [6] if $A \subseteq \tau_2\text{cl}(\tau_1\text{cl}(A))$, $\tau_1\tau_2\text{semi-pre-closed set}$ if $\tau_1\text{int}(\tau_2\text{cl}(\tau_1\text{int}(A))) \subseteq A$.
(vi) $\tau_1\tau_2\eta$-open set [11] if $A \subseteq \tau_1\text{int}(\tau_2\text{cl}((\tau_1\text{cl}(A)))) \cup \tau_2\text{cl}(\tau_1\text{int}(A))$, $\tau_1\tau_2\eta$-closed set if $\tau_1\text{cl}(\tau_2\text{cl}(\tau_1\text{cl}(A))) \cap \tau_2\text{int}(\tau_1\text{cl}(A)) \subseteq A$.

Definition : 2.2

A subset $A$ of a topological space $(X, \tau_1, \tau_2)$ is called

(i) $\tau_1\tau_2\text{g-closed set}$ [4] if $\tau_2\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\tau_1\text{open}$ in $(X, \tau)$.
(ii) $\tau_1\tau_2\text{g*-closed set}$ [10] if $\tau_2\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\tau_1\text{g-open}$ in $(X, \tau)$.
Example 3.7: The converse of the above theorem is not true as shown in the following example.

Proof: Let A be any \( \tau_2 \) closed set in \((X, \tau_1, \tau_2)\) and \( A \subseteq U \), where U is \( \tau_1 \) open. Since every \( \tau_2 \) closed set is \( \tau_2 \eta \)-closed, \( \tau_2 \eta cl(A) \subseteq \tau_2 cl(A) = A \). Therefore \( \tau_2 \eta cl(A) \subseteq A \subseteq U \). Hence A is \( \tau_1 \tau_2 \eta \)-closed set.

The converse of the above theorem is not true as shown in the following example.

Example 3.8: Let \( X = \{a, b, c, d\} \) with \( \tau_1 = \{X, \phi, \{a\}, \{b, d\}, \{a, b, d\}\} \) and \( \tau_2 = \{X, \phi, \{b\}, \{c, d\}, \{b, c, d\}\} \). The set \( \{b\} \) is \( \tau_1 \tau_2 \eta \)-closed but not \( \tau_2 \eta \)-closed.

Theorem 3.9: Every \( \tau_2 \) semi-closed set is \( \tau_1 \tau_2 \eta \)-closed set.

Proof: Let A be any \( \tau_2 \) semi-closed set in \((X, \tau_1, \tau_2)\) and \( A \subseteq U \), where U is \( \tau_1 \) open. Since every \( \tau_2 \) semi-closed set is \( \tau_2 \eta \)-closed, \( \tau_2 \eta cl(A) \subseteq \tau_2 cl(A) = A \). Therefore \( \tau_2 \eta cl(A) \subseteq A \subseteq U \). Hence A is \( \tau_1 \tau_2 \eta \)-closed set.

The converse of the above theorem is not true as shown in the following example.

Example 3.10: Let \( X = \{a, b, c, d\} \) with \( \tau_1 = \{X, \phi, \{a\}, \{b, d\}, \{a, b, d\}\} \) and \( \tau_2 = \{X, \phi, \{b\}, \{c, d\}, \{b, c, d\}\} \). The set \( \{c\} \) is \( \tau_1 \tau_2 \eta \)-closed but not \( \tau_2 \eta \)-closed.

Theorem 3.11: Every \( \tau_2 \) regular-closed set is \( \tau_1 \tau_2 \eta \)-closed set.

Proof: Let A be any \( \tau_2 \) regular-closed set in \((X, \tau_1, \tau_2)\) and \( A \subseteq U \), where U is \( \tau_1 \) open. Since every \( \tau_2 \) regular closed set is \( \tau_2 \) closed. By theorem 3.2, A is \( \tau_1 \tau_2 \eta \)-closed set.

The converse of the above theorem is not true as shown in the following example.

Example 3.12: Let \( X = \{a, b, c, d\} \) with \( \tau_1 = \{X, \phi, \{a\}, \{b, d\}, \{a, b, d\}\} \) and \( \tau_2 = \{X, \phi, \{b\}, \{c, d\}, \{b, c, d\}\} \). The set \( \{d\} \) is \( \tau_1 \tau_2 \eta \)-closed but not \( \tau_2 \) regular closed.
Theorem 3.12: Every $\tau_2g$-closed set is $\tau_1\tau_2g\eta$-closed set.

Proof: Let $A$ be any $\tau_2g$-closed set in $(X, \tau_1, \tau_2)$, then $\tau_2\text{cl}(A) \subseteq U$ whenever $A \subseteq U$, where $U$ is $\tau_1$ open. Since every $\tau_2g$-closed set is $\tau_2\eta$-closed, $\tau_2\eta\text{cl}(A) \subseteq \tau_2\text{cl}(A) = A$. Hence $A$ is $\tau_1\tau_2g\eta$-closed set.

The converse of the above theorem is not true as shown in the following example.

Example 3.13: Let $X = \{a, b, c, d\}$ with $\tau_1 = \{X, \varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ and $\tau_2 = \{X, \varnothing, \{a\}, \{b, d\}, \{a, b, d\}\}$. The set $\{b\}$ is $\tau_1\tau_2g\eta$-closed but not $\tau_2g$-closed.

Theorem 3.14: Every $\tau_2g^*\eta$-closed set is $\tau_1\tau_2g\eta$-closed set.

Proof: Let $A$ be any $\tau_2g^*\eta$-closed set in $(X, \tau_1, \tau_2)$, then $\tau_2\text{cl}(A) \subseteq U$ whenever $A \subseteq U$, where $U$ is $\tau_1$ open. Since every $\tau_2g^*\eta$-closed set is $\tau_2g\eta$-closed. By theorem 3.12 $A$ is $\tau_1\tau_2g\eta$-closed set.

The converse of the above theorem is not true as shown in the following example.

Example 3.15: Let $X = \{a, b, c, d\}$ with $\tau_1 = \{X, \varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ and $\tau_2 = \{X, \varnothing, \{a\}, \{b, d\}, \{a, b, d\}\}$. The set $\{a\}$ is $\tau_1\tau_2g\eta$-closed but not $\tau_2g^*\eta$-closed.

Theorem 3.16: Every $\tau_2g^*\eta$-closed set is $\tau_1\tau_2g\eta$-closed set.

Proof: Let $A$ be any $\tau_2g^*\eta$-closed set in $(X, \tau_1, \tau_2)$, then $\tau_2\text{scl}(A) \subseteq U$ whenever $A \subseteq U$, where $U$ is $\tau_1$ semi-open. Since every $\tau_2$ semi-closed set is $\tau_2\eta$-closed, $\tau_2\eta\text{cl}(A) \subseteq \tau_2\text{scl}(A) \subseteq U$. Hence $A$ is $\tau_1\tau_2g\eta$-closed set.

The converse of the above theorem is not true as shown in the following example.

Example 3.17: Let $X = \{a, b, c, d\}$ with $\tau_1 = \{X, \varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ and $\tau_2 = \{X, \varnothing, \{a\}, \{b, d\}, \{a, b, d\}\}$. The set $\{a\}$ is $\tau_1\tau_2g\eta$-closed but not $\tau_2g^*\eta$-closed.

Theorem 3.18: Every $\tau_2g\eta$-closed set is $\tau_1\tau_2g\eta$-closed set.

Proof: Let $A$ be any $\tau_2g\eta$-closed set in $(X, \tau_1, \tau_2)$, then $\tau_2\text{acl}(A) \subseteq U$ whenever $A \subseteq U$, where $U$ is $\tau_1$ open. Since every $\tau_2g\eta$-closed set is $\tau_2\eta$-closed, $\tau_2\eta\text{cl}(A) \subseteq \tau_2\text{acl}(A) \subseteq U$ and hence $A$ is $\tau_1\tau_2g\eta$-closed set.

The converse of the above theorem is not true as shown in the following example.

Example 3.19: Let $X = \{a, b, c, d\}$ with $\tau_1 = \{X, \varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ and $\tau_2 = \{X, \varnothing, \{a\}, \{b, d\}, \{a, b, d\}\}$. The set $\{a\}$ is $\tau_1\tau_2g\eta$-closed but not $\tau_2g\eta$-closed.

Theorem 3.20: Every $\tau_2g\alpha$-closed set is $\tau_1\tau_2g\eta$-closed set.

Proof: Let $A$ be any $\tau_2g\alpha$-closed set in $(X, \tau_1, \tau_2)$, then $\tau_2\text{acl}(A) \subseteq U$ whenever $A \subseteq U$, where $U$ is $\tau_1$ open. Since every $\tau_2g\alpha$-closed set is $\tau_2\eta$-closed, $\tau_2\eta\text{cl}(A) \subseteq \tau_2\text{acl}(A) \subseteq U$. Hence $A$ is $\tau_1\tau_2g\eta$-closed set.

The converse of the above theorem is not true as shown in the following example.

Example 3.21: Let $X = \{a, b, c, d\}$ with $\tau_1 = \{X, \varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ and $\tau_2 = \{X, \varnothing, \{a\}, \{b, d\}, \{a, b, d\}\}$. The set $\{a\}$ is $\tau_1\tau_2g\eta$-closed but not $\tau_2g\alpha$-closed.

Remark 3.22: Let $A$ and $B$ be two $\tau_1\tau_2$ closed sets, then their union and intersection need not be $\tau_1\tau_2 g\eta$-closed as shown from the following examples.

Example 3.23: Let $X = \{a, b, c, d\}$ with $\tau_1 = \{X, \varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ and $\tau_2 = \{X, \varnothing, \{a\}, \{b, d\}, \{a, b, d\}\}$. Here the set $\{a\}$, $\{b\}$ are $\tau_1\tau_2 g\eta$-closed sets and $\{a\} \cup \{b\}$ is not $\tau_1\tau_2 g\eta$-closed.

Example 3.24: Let $X = \{a, b, c, d\}$ with $\tau_1 = \{X, \varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ and $\tau_2 = \{X, \varnothing, \{a\}, \{b, d\}, \{a, b, d\}\}$. Here $\{a, b\} \cap \{a, b\}$ is $\tau_1\tau_2 g\eta$-closed sets and $\{a, b, c\} \cap \{a, b, d\}$ is not $\tau_1\tau_2 g\eta$-closed.

Theorem 3.25: Let $A$ be a subset of a bitopological space $(X, \tau_1, \tau_2)$. If $A$ is $\tau_1\tau_2 g\eta$-closed, $\tau_2\eta\text{cl}(A) = \text{Adoes}$ not contain any non-empty $\tau_1$ closed set.
Theorem 4.2: 
Theorem 3.27: 

Proof: 

Corollary 3.26: 

Conversely, suppose that A is $\tau_1 \tau_2$ g$n$-closed and $\tau_2$ closed. Since A is $\tau_2$ closed, we have $\tau_2 cl(A) = A$. Therefore, $\tau_2 cl(A) - A = \emptyset$ which is $\tau_1 \tau_2$.

Theorem 3.27: Let A and B be subsets of a bitopological space $(X, \tau_1, \tau_2)$, such that $A \subseteq B \subseteq \tau_2 cl(A)$. If A is $\tau_1 \tau_2$ g$n$-closed, then B is also $\tau_1 \tau_2$ g$n$-closed.

Proof: Let $B \subseteq P$ and $P$ is $\tau_1$ open in X. Since $A \subseteq B$, we have $A \subseteq P$. Since A is $\tau_1 \tau_2$ g$n$-closed, we have $\tau_2 cl(A) \subseteq P$. As $B \subseteq \tau_2 cl(A)$, $\tau_2 cl(B) \subseteq \tau_2 cl(A)$. Hence $\tau_2 cl(B) \subseteq P$. Therefore B is $\tau_1 \tau_2$ g$n$-closed.

IV. $\tau_1 \tau_2$ G$H$-OPEN SETS IN BITOPOLOGICAL SPACES

Definition 4.1: A subset A of $(X, \tau_1, \tau_2)$ is said to be $\tau_1 \tau_2$ g$n$-open in X if its complement $X - A$ is $\tau_1 \tau_2$ g$n$-closed in $(X, \tau_1, \tau_2)$.

Theorem 4.2: A subset A of a bitopological space $(X, \tau_1, \tau_2)$ is $\tau_1 \tau_2$ g$n$-open if and only if $P \subseteq \tau_2 int(A)$ whenever $P \subseteq A$ and P is $\tau_1$ closed in X.

Proof: Let A is $\tau_1 \tau_2$ g$n$-open. Let $P \subseteq A$ and $P_{\tau_1} \tau_2$ closed in X. Then $A' \subseteq P'$ and $P'$ is $\tau_1$ open in X. Since A is $\tau_1 \tau_2$ g$n$-open, we have $A' = \tau_1 \tau_2$ g$n$-closed. Hence $\tau_2 cl(A') \subseteq P'$. Since $\tau_2 cl(A) = (\tau_2 int(A))'$, consequently, $(\tau_2 int(A))' \subseteq P$. Therefore $P \subseteq \tau_2 int(A)$.

Conversely, suppose that $P \subseteq \tau_2 int(A)$ whenever $P \subseteq A$ and $P_{\tau_1} \tau_2$ closed in X. Let $A' \subseteq Q$ and $Q_{\tau_1} \tau_2$ open in X. Then $Q' \subseteq A$ and $Q'$ is $\tau_1$ closed in X. By hypothesis, $Q' \subseteq \tau_2 int(A)$. That is, $(\tau_2 int(A))' \subseteq Q$. Therefore $\tau_2 cl(A') \subseteq Q$. Consequently $A'$ is $\tau_1 \tau_2$ g$n$-closed. Hence A is $\tau_1 \tau_2$ g$n$-open.

Theorem 4.3: Let A and B be subsets of a bitopological space $(X, \tau_1, \tau_2)$ such that $\tau_2 int(A) \subseteq B \subseteq A$. If A is $\tau_1 \tau_2$ g$n$-open, then B is also $\tau_1 \tau_2$ g$n$-open.

Proof: Suppose that A and B are subsets of a bitopological space $(X, \tau_1, \tau_2)$ such that $\tau_2 int(A) \subseteq B \subseteq A$, let A be $\tau_1 \tau_2$ g$n$-open. Then $A' \subseteq B' \subseteq \tau_2 cl(A')$. Since $A'$ is $\tau_1 \tau_2$ g$n$-closed. By theorem 3.27, $B' \tau_1 \tau_2$ g$n$-closed in X. Therefore B is $\tau_1 \tau_2$ g$n$-open.

V. $\tau_1 \tau_2$ G$H$-CLOSURE OF A SETS IN BITOPOLOGICAL SPACES

Definition 5.1: For a subset A of $(X, \tau_1, \tau_2)$, the intersection of all $\tau_1 \tau_2$ g$n$-closed sets containing A is called the $\tau_1 \tau_2$ g$n$-closure of A and is denoted by $\tau_1 \tau_2$ g$n$-cl(A).

That is, $\tau_1 \tau_2$ g$n$-cl(A) = $\cap \{ M : A \subseteq M, M \ is \ \tau_1 \tau_2$ g$n$-closed in X \}.

Remark: 5.2 If A and B are any two subsets of a bitopological space $(X, \tau_1, \tau_2)$, then $\tau_1 \tau_2$ g$n$-cl(X) = $\tau_1 \tau_2$ g$n$-cl($\emptyset$) = $\emptyset$.

Example 5.3 Let X = {a, b, c, d} with $\tau_1$ = {X, $\emptyset$, {a}, {b}, {a, b}, {a, b, c}} and $\tau_2$ = {X, $\emptyset$, {a}, {b, d}, {a, b, d}}. $\tau_1 \tau_2$ g$n$-closed sets are {X, $\emptyset$, {a}, {b}, {c}, {d}, {a, c}, {a, d} {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}}. Let A = {a, c}. $\tau_1 \tau_2$ g$n$-cl(A) = {a, c}, $\tau_1 \tau_2$ g$n$-cl(X) = X, $\tau_1 \tau_2$ g$n$-cl(\$\emptyset$) = $\emptyset$.

Remark 5.4 If A and B are any two subsets of a bitopological space $(X, \tau_1, \tau_2)$, then

(i) A $\subseteq$ B $\Rightarrow$ $\tau_1 \tau_2$ g$n$-cl(A) $\subseteq$ $\tau_1 \tau_2$ g$n$-cl(B).
Remark 6.4:

Proof: Let N be a neighbourhood of a point x if and only if there exist an open set G such that x ∈ G and x ∉ F.

⇒ There exists a τ₁τ₂ -open set F containing x and x ∉ F.

⇒ There exists a τ₁τ₂ -open set X–F containing x and A ∩ (X–F) = φ which is a contradiction. Therefore x ∈ τ₁τ₂ -cl(A).

VI. τ₁τ₂GH-NBNEIGHBOURHOODS IN BITOPOLITICAL SPACES

Definition 6.1: Let X be a topological space and let x ∈ X. A subset N of X is said to be a τ₁τ₂ -neighbourhood of x if and only if there is a τ₁τ₂ -open set G such that x ∈ G ⊆ N.

Definition 6.2: A subset N of a space X, is called a τ₁τ₂ -neighbourhood of A ⊆ X if and only if there exists a τ₁τ₂ -open set G such that A ⊆ G ⊆ N.

Theorem 6.3: Every neighbourhood N of x ∈ X is a τ₁τ₂ -neighbourhood of (X, τ₁, τ₂).

Proof: Let N be a neighbourhood of a point x ∈ X. To prove that N is a τ₁τ₂ -neighbourhood of x. By definition 6.2, there exist an open set G such that x ∈ G ⊆ N. As every open set is a τ₁τ₂ -open set G such that x ∈ G ⊆ N. Hence N is a τ₁τ₂ -neighbourhood of x.

Remark 6.4: In general a τ₁τ₂ -neighbourhood N of x ∈ X need not to be a neighbourhood of x in X, as in the following example.

Example 6.5: Let X = {a, b, c, d} with τ₁ = {X, φ, {a}, {b}, {a, b}, {a, b, c}} and τ₂ = {X, φ, {a}, {b, d}, {a, b, d}}. The set {a, c} is a τ₁τ₂ -neighbourhood of the point c, since the τ₁τ₂ -open set {a, c} is such that c ∈ {a, c} ⊆ {a, c}. However the set {a, c} is not a neighbourhood of the point {b}, since no open set G exists such that c ∈ G ⊆ {a, c}.

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